# VIII Whitney's theorem for maps between two-manifolds

Let $m = n = 2$ and $f \in C^\infty(M, N)$ one-generic

$\Rightarrow$ (Thm II.8)

- $\Sigma^1(f)$ is one-dim submf of $M$
- $\Sigma^i(f) = \emptyset$ for $i \geq 2$.

Cor VI.8 :

- if $f$ two-generic, then $\Sigma^{1,1}(f)$ is $0$-dim submf and no other cases occur

- $x \in \Sigma^1(f)$ then either

  a) $T_x \Sigma^1(f) \oplus \ker df(x) = T_x M$

  $\Rightarrow x \in \Sigma^{1,0}(f)$

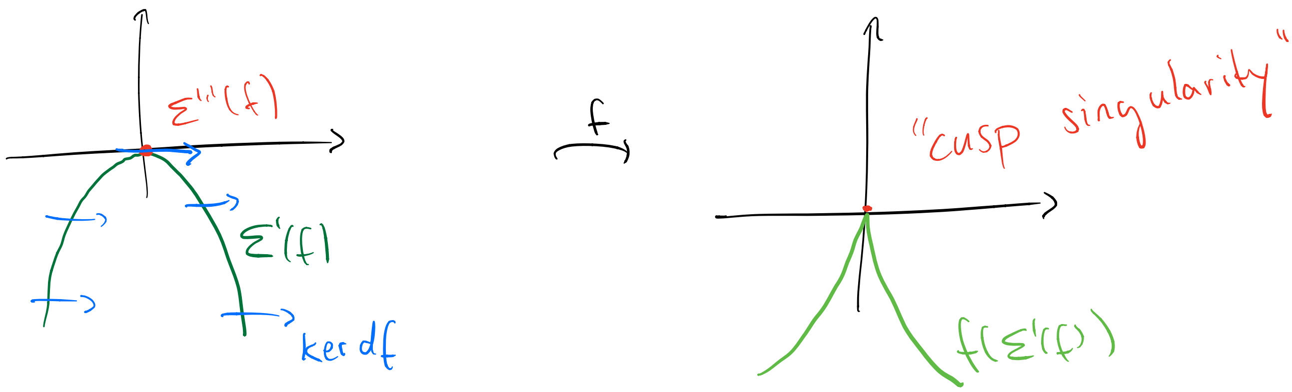
  b) $T_x \Sigma^1(f) = \ker df(x) \Rightarrow x \in \Sigma^{1,1}(f)$

(cf. exercises 5.4 and 2.2)

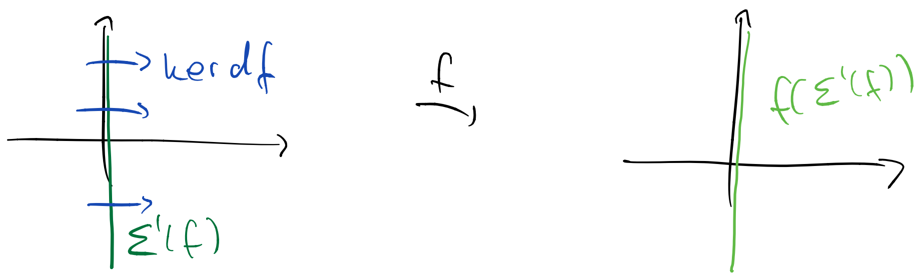
e.g

• <u>pleat</u> $f : (x,y) \mapsto (x^3 + xy, y)$

$$df(x,y) = \begin{pmatrix} 3x^2+y & x \\ 0 & 1 \end{pmatrix}$$




• <u>fold</u> $f : (x,y) \mapsto (x^2, y)$ $df(x,y) = \begin{pmatrix} 2x & 0 \\ 0 & 1 \end{pmatrix}$




Whitney showed that these two examples are in fact the only stable/generic cases,

a) fold, b) pleat/cusp.

1. **Theorem:** If a) holds, then $x$ is a fold point, i.e. there exist coordinates around $x$ and $f(x)$, s.t. $f$ is locally given by

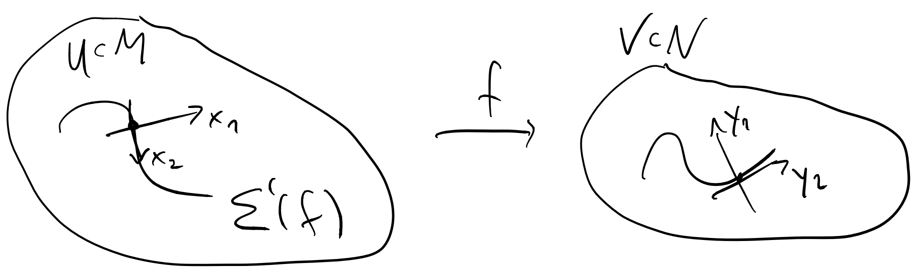
$$(x_1, x_2) \longmapsto (x_1^2, x_2)$$

**Proof:**

a) $\Rightarrow$ $f|_{\Sigma^1(f)}$ is immersion at $x$, so locally a diffeom. Choose coordinates

$(y_1, y_2)$ s.t. $f(\Sigma^1(f)) \overset{loc.}{=} \{y_1 = 0\}$

and $(x_1, x_2)$ s.t. $x_2 = y_2 \circ f(x_1, x_2)$, $\Sigma^1(f) \overset{loc.}{=} \{x_1 = 0\}$




So, loc. $f$ is given by $(x_1, x_2) \mapsto (h(x_1, x_2), x_2)$.

From 1. $\Sigma^1(f) = \{x_1 = 0\}$ and also $\Sigma^1(f) = \left\{ \frac{\partial h}{\partial x_1} = 0 \right\}$

2. $f(\Sigma^1(f)) = \{y_1 = 0\}$ $\Rightarrow$ $h|_{x_1 = 0} = 0$

3. $\left.\frac{\partial^2 h}{\partial x_1^2}\right|_{x_1=0} \neq 0$ (bec. $f$ is one-generic, $j^1 f \pitchfork S^1$ in $J^1(M,N)$)

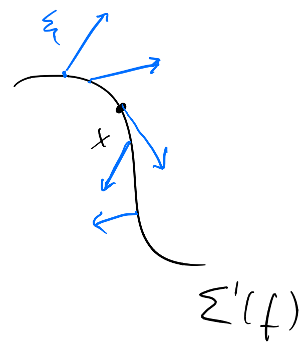
We conclude $h(x_1, x_2) = x_1^2$.

Now case b), $T_x \Sigma^1(f) = \ker df(x)$.

Let $\xi$ be a vector field along $\Sigma^1(f)$ with $\xi \neq 0$ and $\xi(x') \in \ker df(x')$ $\forall x' \in U_x \cap \Sigma^1(f)$.

Let $\Sigma^1(f)$ locally be given by $\{k = 0\}$ with $k$ smooth, $dk \neq 0$.

We consider the function




$$K : \Sigma^1(f) \to \mathbb{R} \qquad x' \mapsto dk(x') \cdot \xi(x')$$

which vanishes at $x$.

2. Def.: $x$ is a simple cusp if it is a simple zero of $K$.

Exerc.: Show this defn is indep. of the choice $(\xi, k)$.

**3. Thm:** If $x$ is a simple cusp, then there are coordinates around $x$ and $f(x)$ such that $f$ is locally given by

$$(x_1, x_2) \longmapsto (x_1^3 + x_1 x_2, x_2).$$

**Proof:**

As in the proof of Thm 1 choose coord. $(x_1, x_2)$ and $(y_1, y_2)$ s.t.

$$f: (x_1, x_2) \mapsto (h(x_1, x_2), x_2)$$

and
$$df(x_1, x_2) = \begin{pmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \\ 0 & 1 \end{pmatrix} \underset{\text{at } (0,0)}{\overset{\uparrow}{=}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

i.e. $$\frac{\partial h}{\partial x_1}(0,0) = \frac{\partial h}{\partial x_2}(0,0) = 0$$

but $$d\left(\frac{\partial h}{\partial x_1}\right)(0,0) \neq 0 \quad (\text{bec. } f \text{ is one-generic})$$

$$\underbrace{\Sigma^1(f) = \left\{ \frac{\partial h}{\partial x_1} = 0 \right\}}_{= \text{"}k\text{"}} \quad \text{and} \quad \underbrace{\ker df = \left\langle \frac{\partial}{\partial x_1} \right\rangle}_{= \text{"}\xi\text{"}},$$

"$k$" ⟵ in Def. 2 ⟶ "$\xi$"

so if $x$ is a simple cusp, then

$$\frac{\partial^2 h}{\partial x_1^2}(0,0) = 0 \quad \text{and} \quad \frac{\partial^3 h}{\partial x_1^3}(0,0) \neq 0 .$$

Thus, at the origin: $\quad h = \frac{\partial h}{\partial x_1} = \frac{\partial^2 h}{\partial x_1^2} = 0$

$$\frac{\partial^3 h}{\partial x_1^3} \neq 0$$

Malgrange preparation thm:

$$h(x_1, x_2) = c(x_1, x_2)\left( x_1^3 + a_2(x_2)\, x_1^2 + a_1(x_2)\, x_1 + a_0(x_2) \right)$$

with $c$ and the $a_i$ smooth, $c(0,0) \neq 0$.

$$\Leftrightarrow \quad x_1^3 = \gamma(h, x_2) + \beta(h, x_2) \cdot x_1 + \alpha(h, x_2) \cdot x_1^2$$

$$\Leftrightarrow \quad (x_1 - A)^3 + B \cdot (x_1 - A) = C \qquad \circledast$$

with $A, B, C : \mathbb{R}^2 \to \mathbb{R}$ smooth, vanishing at $(0,0)$.

1. Set $x_2 = 0$. $*$ becomes

$$\left(x_1 - A(h(x_1,0),0)\right)^3 + B(h(x_1,0), 0) \cdot \left( x_1 - A(h(x_1,0),0) \right) = C\left(h(x_1,0), 0\right)$$

Since the leading order of $h(x_1, 0)$ is $x_1^3$,

the LHS of $*$ is $x_1^3$ + higher order terms.

The RHS of $*$ can only match this if

$$\partial_1 C(0,0) \neq 0.$$

2. Expand $A, B, C$ to first order around $(0,0)$. Then $*$ gives

$$(x_1 - \partial_1 A \cdot h - \partial_2 A \cdot x_2)^3 + (\partial_1 B \cdot h + \partial_2 B \, x_2) \cdot$$
$$(x_1 - \partial_1 A \cdot h - \partial_2 A \cdot x_2) = \partial_1 C \cdot h + \partial_2 C \cdot x_2 .$$

"First" nonzero derivative of $h$ at $(0,0)$ is $\frac{\partial^2 h}{\partial x_1 \partial x_2}$,

so $h \propto x_1 x_2 + x_2^2 + \dots$

$$\Bigg\} \Rightarrow (x_1 - \partial_1 A \cdot (x_1 x_2 + \dots) - \partial_2 A \cdot x_2)^3$$
$$+ (\partial_1 B \cdot (x_1 x_2 + \dots) + \partial_2 B \cdot x_2) \cdot (x_1 - \partial_1 A \cdot (x_1 x_2 + \dots) - \partial_2 A \cdot x_2)$$
$$= \partial_1 C \cdot (x_1 x_2 + \dots) + \partial_2 C \cdot x_2$$

$$\Rightarrow \quad \partial_2 B(0,0) = \underline{\partial_1 C(0,0) \overset{1.}{\neq} 0}$$
and $\underline{\partial_2 C(0,0) = 0}$ (by comparing linear and quadratic terms)

Now define new coordinates by

- $(\bar{x}_1, \bar{x}_2) = \phi(x_1, x_2) := (x_1 - A(h, x_2),\ B(h, x_2))$

$$d\phi(0,0) = \begin{pmatrix} 1 - \partial_1 A(0,0) \cdot \underbrace{\partial_1 h(0,0)}_{=0} & -\partial_1 A(0,0)\, \underbrace{\partial_2 h(0,0)}_{=0} - \partial_2 A(0,0) \\ \partial_1 B(0,0) \cdot \underbrace{\partial_1 h(0,0)}_{=0} & \partial_2 B(0,0) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\partial_2 A(0,0) \\ 0 & \neq 0 \end{pmatrix}$$

$\Rightarrow$ loc. diffeom.

- $(\bar{y}_1, \bar{y}_2) = \psi(y_1, y_2) := (C(y_1, y_2),\ B(y_1, y_2))$

$$d\psi(0,0) = \begin{pmatrix} \partial_1 C(0,0) & \partial_2 C(0,0) \\ \partial_1 B(0,0) & \partial_2 B(0,0) \end{pmatrix} = \begin{pmatrix} \neq 0 & 0 \\ \partial_1 B(0,0) & \neq 0 \end{pmatrix}$$

$\Rightarrow$ loc. diffeom.

Finally, $\psi \circ f \circ \phi^{-1} : (\bar{x}_1, \bar{x}_2) \xmapsto{\phi^{-1}} (x_1, x_2) \xmapsto{f} (h(x_1, x_2), x_2)$

$$\downarrow \psi$$

$$(C(h, x_2),\ B(h, x_2))$$

$$= \left( (x_1 - A(h,x_2))^3 + B(h,x_2)\cdot(x_1 - A(h,x_2)) \, , \, B(h,x_2) \right)$$

$$= \left( \overline{x}_1^{\,3} + \overline{x}_2 \cdot \overline{x}_1 \, , \, \overline{x}_2 \right).$$

**4. Theorem:** There is a residual set in $C^\infty(M,N)$ s.t. if $f$ belongs to this set, then its only singularities are folds and simple cusps.

Proof:

The sought-after set is given by the two-generic maps in $C^\infty(M,N)$.

1. Being two-generic is determined by $j^3 f$
   ( $j^2 f \pitchfork S^{ij}$ is a condition on $j^3 f$ ! )

2. $f$ two-generic $\Rightarrow$ $\dim \Sigma^1(f) = 1$
   $\dim \Sigma^{1,1}(f) = 0$
   and no other singularities possible.

3. If $f$ two-generic and has a cusp singularity, then it must be simple:

Suppose $f : (x_1, x_2) \mapsto ( h(x_1,x_2), x_2 )$

with $\frac{\partial^k h}{\partial x_1^k} = 0$ for $k = 0, \dots, 3$

Then define $\tilde{f} : (x_1, x_2) \mapsto (x_2 \cdot \alpha(x_1, x_2), x_2)$

with

$$\alpha(x_1, x_2) = \frac{\partial_{222} h(0)}{6} x_2^2 + \frac{\partial_{22} h(0)}{2} x_2$$
$$+ \partial_2 h(0) + \frac{\partial_{112} h(0)}{2} x_1^2 + \partial_{12} h(0) \cdot x_1$$
$$+ \frac{\partial_{122} h(0)}{2} x_1 \cdot x_2 .$$

$\tilde{f}$ has the same 3-jet as $f$ at $(0,0)$, so by 1. also two-generic, but

$d\tilde{f} = \begin{pmatrix} x_2 \cdot \partial_1 \alpha & \cdots \\ 0 & 1 \end{pmatrix}$, so $\Sigma^{1,1}(f)$ is 1-dim. $\lightning$

$\blacksquare$

If $n = 2$, but $m \geq 3$, then we still find $\dim \Sigma^1(f) = 1$, $\dim \Sigma^{1,1}(f) = 0$ (in the generic case). However, there are more types of folds and cusps possible.

For $N = \mathbb{R}^2$ the situation stays quite simple, allowing to study the source manifold $M$ a la Morse theory.

A starting point is the following observation:

<u>S. Prop:</u> Let $M$ compact and $f: M \to \mathbb{R}^2$ smooth & two-generic. Then $\Sigma^1(f)$ is a finite disjoint union of $S^1$'s and $\Sigma^{1,1}(f)$ is a finite collection of points, and no other singularities occur.

<u>Proof:</u> Exercise.

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